Sound Field Simulation and Analysis Using Wave Equation and Finite Differences Method

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***Abstract*—This paper presents a sound field analysis method which uses numerical solving of the wave equation. The wave equation is described, its analytical and numerical solutions, and its application in sound field analysis. Finite difference method is used for solving the equation. Comparison of analytical and numerical solutions is shown, together with a discussion on convergence and stability of the numerical solution. Finally, the sound field simulator based on numerical solving of wave equation is presented. Applicability of sound field simulator for developing object detection algorithms is also analyzed.**

***Keywords*—sound field, wave equation, finite differences method, sound field simulator**

I. Introduction

Many localization problems, detection of objects and/or its position in an unstructured environment uses some kind of optical sensors, sonar, radar, GPS, cameras or lasers (as in [1],[2]). Most of the research in this area today is based on Image processing, and combined sensing techniques, primarily insipred by the ways that humans and animals move in unknown environments. Humans mostly use their vision to get informations from the world, but hearing, and getting informations from sounds of the environment is often underestimated and not exploited enough. However, there are many examples from nature (e.g. bats), which show that the biological acoustic system has great potential in localization and object detection.

The authors of this paper are trying to investigate the possibility of syntesis of an (ultra)sound sensoring system that would be capable of solving the localization problem in unstructured environments, or object detection in a known sound field. An object positioned in a known field (in this case a sound field) will undoubtedly affect the characteristics of the field itself, and change its properties and behaviour. The possibility of partial or complete reconstruction of the sound field by measuring it at several points is a challenging subject of research. As it is going to be shown in the paper, testing this hipothesis in the sound field simulator, has proven that this is partially possible.

Sound is usually defined as a travelling wave which is an oscillation of pressure transmitted through a medium. The problem with a sound transducer or transciever is that they are generally very directional, so spatial resolution must be realized with fast space scanning.

Sound field analysis on its physical level is used in this project. Even though a mathematical description of the sound field looks fairly simple (described by the linear partial differential equation), finding an analytical solution of a sound field is not trivial in case of general initial and boundary conditions. Consenquently, for the acoustic analysis of the sound field, and to get test data, it was necessary to analyze the wave equation with some numerical method. A sound field simulator, based on numerical finite differences method of solving a partial differential equation is developed. Some of the results of this simulator are shown in this paper.

When numerically solving a continual analytical problems, some approximations are inevitable. Because these approximations can lead to instability of the solution, part of this paper deals with errors, stability and disadvantages of the numerical approach.

This paper will analyse sound fields in air only, but most of the discussion can be applied for other fluids and configurations as well.

II. The Wave Equation

Sound waves are longitudinal waves, and they propagate through space using compression and rarefaction on the medium, on the direction of wave propagation. Medium particles (mostly in a sinusoidal manner) oscillate around equillibrium position in direction of the wave, causing medium movement. Besides particle movement, sound can be described with sound pressure, particle velocity, sound intensity, acoustic impedance, speed of sound, etc.

There are several approaches which lead to the description of the sound field, the so called wave equation. The wave equation is the prototypical example of a hyperbolic partial differential equation [3]. In its simplest form, the wave equation refers to a scalar function $u = (x_1, x_2, ..., x_n, t) = u(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, that satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad (1)$$

where ∇^2 is the (spatial) Laplacian and where c is a fixed constant equal to the propagation speed of the wave. This is known as the non-dispersive wave equation. For a sound wave in air at $20C$ this constant is about $343[m/s]$. More realistic differential equations for waves allow for the speed of wave propagation to vary with the frequency, temperature, pressure, amplitude of the wave etc. (noted $c = c(*)$) leading to a nonlinear wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c(*)^2 \nabla^2 u \tag{2}$$

A wave may also be superimposed onto another movement (for instance sound propagation in a moving medium like a gas flow). In that case the scalar u will contain a Mach factor.

Some form of wave equation, or its various generalizations, almost inevitably arise in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. In fact, the studies of water waves, acoustic waves, elastic waves in solids, and electromagnetic waves are all based on this equation.

III. Initial and Boundary conditions of wave equation

Wave equation is almost always subject to initial and boundary conditions. Typically for a wave equation initial conditions are given in form:

$$u(\mathbf{x}, 0) = f(\mathbf{x}),\ \frac{\partial u(\mathbf{x}, 0)}{\partial t} = g(\mathbf{x}) \tag{3}$$

The wave equation usually deals with two types of boundary conditions ([3],[4]), Dirichlet (4) and Neumann (5):

$$u(\mathbf{x}, t) = h(\mathbf{x}, t), \forall \mathbf{x} \in \Omega_X \tag{4}$$

$$\frac{\partial u(\mathbf{x}, t)}{\partial n} = h(\mathbf{x}, t), \forall \mathbf{x} \in \Omega_X \tag{5}$$

Here, n denotes the (typically exterior) normal to the boundary Ω_x.

Boundary conditions (BC) are crucial in analysis of sound field with objects that can cause reflection, apsorption, refraction or difraction. Dirichlet BC defines the behaviour of the solution of the wave equation on domain boundaries, while Neumann BC specifies the values that the derivative of a solution is to take on the boundary of the domain. Dirihlet BC is important in determining the sound field in the neighborhood of solid boundaries (e.g. closed rooms, solid obstacles), while Neumann BC is necessary when modeling open spaces (derivation in (5) equals zero), or elastic boundary conditions (see e.g. [5]).

Determining the form of boundary conditions, and solving wave equations taking it in consideration, can be difficult.

IV. Analytical Solution of the Wave Equation

A technique known as the method of separation of variables is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation. Differential equation (1) can also be solved using method of characteristics, Fourier series, method of eigenfunctions, integral transform methods, finite Fourier transforms, etc.[6] It can be shown [6] that the general solution of a 1D wave equation is given with (6), where m and n are arbitrary functions, with m representing a right-traveling wave and n a left-traveling wave.

$$u(x, t) = m(x - ct) + n(x + ct) \tag{6}$$

It can also be shown that the general solution of the initial value problem for a wave equation describing 1D string is given by (7), where $u(x, t = 0) = u_0(x)$, and $\partial u/\partial t|_{(t=0)} = v_0(x)$.

$$u(x, t) = \frac{u_0(x - ct) + u_0(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)ds \tag{7}$$

In analoguous way, but with a fairly complicated procedure, it is possible to derive the analytical solution of wave equations of higher dimensions.

An important result that comes from solving the wave equation is that the general solution is composed from two wave functions - incident wave and a reflected wave. Another interesting characteristic is superposition - existence of one wave in a certain space and time should not affect the existence or properties of another wave in the same space and time.

V. Numerical Solving the Wave Equation

In dealing with many equations arising from the modelling of physical problems, the determination of exact analytical solutions in a simple domain is a formidable task even when the boundary and/or initial data are relatively simple. It is then necessary to resort to numerical or approximation methods in order to deal with problems that cannot be solved analytically.

Most common numerical and approximation approaches to the solution of partial differential equations are numerical methods based on finite difference approximations, variational methods, Rayleigh-Ritz, Galerkin and Kantorovich methods of approximation, etc. [6]

A sound field simulator developed especially for this project is based on Finite Difference Method (FDM). For the sake of simplification,let us assume we are solving a two-dimensional (2D) wave equation (8), where $u(x, y, t)$ is a function of two independent spatial variables, and time. Equation (1) leads to:

$$L(u(x, y, t)) = \frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = 0 \tag{8}$$

Space domain is divided on a set of uniformly spaced rectangles with dimension $\Delta x \times \Delta y$, with vertices at $P_{i,j}$ with coordinates $(i\Delta x, j\Delta y)$, where i, j, are positive or negative integers or zero, as shown in Fig.1. Also, the time domain is divided on a set of segments $t = k\Delta t$, where k is a positive integer or zero. We denote $u(i\Delta x, j\Delta y, k\Delta t)$ by $u^k_{i,j}$.

Using the Taylor series expansion, the nth-order central differences expressions of arbitrary function f with spacing h can be derived [7], and are given by:

$$\delta^n_h[f](x) = \sum_{i=0}^{n}(-1)^i \binom{n}{i} f\left(x + \left(\frac{n}{2} - i\right) h\right) \tag{9}$$

The difference between the exact value of the n-th derivation of the function f, and the n-th difference ($O(h^2)$ for central

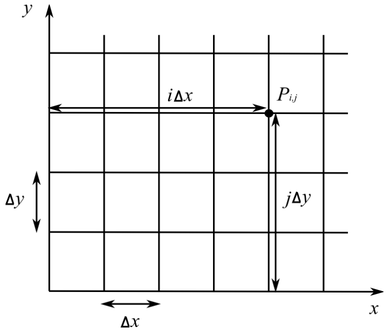

Fig. 1. Uniformly segmented spatial domain

difference) is known as the truncation error in this discretization process. The central difference (9) yields a more accurate approximation than forward and backward differences (which is $O(h)$) [7]. If f is twice continuously differentiable its error is proportional to square of the spacing h^2.

Using (9) it can be easily derived:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{(\Delta x)^2}(u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k) + O((\Delta x)^2) \quad (10)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{(\Delta y)^2}(u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k) + O((\Delta y)^2) \quad (11)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{(\Delta t)^2}(u_{i,j}^{k+1} - 2u_{i,j}^k + u_{i,j}^{k-1}) + O((\Delta t)^2) \quad (12)$$

Substituting (10),(11) and (12) in equation (8), and omissioning truncation errors, equation (8) leads to:

$$F(u_{i,j}^k) = \frac{1}{(\Delta t)^2}(u_{i,j}^{k+1} - 2u_{i,j}^k + u_{i,j}^{k-1}) -$$

$$-c^2(\frac{1}{(\Delta x)^2}(u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k) +$$

$$+\frac{1}{(\Delta y)^2}(u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k)) = 0 \quad (13)$$

Suppose $u(x, y, t)$ represents the exact solution of a partial differential equation $L(u(x, y, t)) = 0$, and $u_{i,j}^k$ is the exact solution of the corresponding finite difference equation $F(u_{i,j}^k) = 0$. Then, the finite difference scheme is said to be convergent if $u_{i,j}^k$ tends to $u(x, y, t)$ as Δx, Δy and Δt tend to zero. The difference, $d_{i,j}^k = (u(i\Delta x, j\Delta y, k\Delta t) - u_{i,j}^k)$ is called the cummulative truncation (or discretization) error. This error can generally be minimized by decreasing the grid sizes.

Another kind of error is introduced when a partial differential equation is approximated by a finite difference equation. If the exact finite difference solution $u_{i,j}^k$ is replaced by the exact solution $u(i\Delta x, j\Delta y, k\Delta t)$ of the partial differential equation at the grid points $P_{i,j}$, then the value $F(u(i\Delta x, j\Delta y, k\Delta t))$ is called the local truncation error at $P_{i,j}$. The finite difference scheme and the partial differential equation are said to be consistent if $F(u(i\Delta x, j\Delta y, k\Delta t))$ tends to zero as Δx, Δy, Δt tends to zero.

In general, finite difference equations cannot be solved exactly because the numerical computation is carried out only up to a finite number of decimal places. Consequently, another kind of error is introduced in the finite difference solution during the actual process of computation. This kind of error is called the round-off error. In practice, the actual computational solution is $u_{i,j}^{k*}$, but not $u_{i,j}^k$, so that the difference $r_{i,j}^k = u_{i,j}^k - u_{i,j}^{k*}$ is the round-off error at the grid point $P_{i,j}$. In fact, this error is introduced into the solution of the finite difference equation by round-off errors. In contrast to the cummulative truncation error, the round-off error cannot be made small by allowing the grid spacing to tend to zero. Thus, the total error involved in the finite difference analysis at the point $P_{i,j}$ is given by $u(i\Delta x, j\Delta y, k\Delta t) - u_{i,j}^{k*} = d_{i,j}^k - r_{i,j}^k$.

Usually the discretization error $d_{i,j}^k$ is bounded when $u_{i,j}^k$ is bounded. This fact is used or assumed in order to introduce the concept of stability. The finite difference algorithm is said to be stable if the round-off errors are sufficiently small, that is, the growth of $r_{i,j}$ can be controlled. Taking into consideration the stability of iterational process (13), and Lax Equivalence theorem [7], the convergence of the solution of (13) can be proven.

VI. Comparation of Numerical and Analytical Results

Equation (13) can be effectively used as a computational algorithm for 2D sound field simulations for numerically solving the wave equation. Similar finite difference schemes can be developed for 1D and 3D problems. An illustration of the simulator computations using the following example is below.

The simulator results of the following wave equation ($c = 1$) will be demonstrated:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \quad (14)$$

with the boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t > 0 \quad (15)$$

and the initial conditions

$$u(x, 0) = sin\pi x, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad 0 \le x \le 1. \quad (16)$$

in several points of space and time domain.

Analytical solution of the problem is $u(x, t) = cos\pi t sin\pi x$, so it is possible to calculate the real values of the solution by simple substitution. Using an equation similar to (13), it is possible to calculate the solutions of (14) in discrete space and time points. Both numerically and analytically calculated results are shown in Table I. A comparison of the analytical solutions with the finite difference solutions calculated with a simulator programmed in the way described shows that the latter results are very accurate.

TABLE I
NUMERICALLY AND ANALYTICALLY CALCULATED RESULTS

Numerical sol.	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$x = 0.1$	0.2940	0.5593	0.7698	0.9050
$x = 0.2$	0.2503	0.4761	0.6553	0.7703
$x = 0.3$	0.1820	0.3462	0.4766	0.5602
$x = 0.4$	0.0960	0.1825	0.2512	0.2953
Analytical sol.	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$x = 0.1$	0.2939	0.5590	0.7694	0.9045
$x = 0.2$	0.2500	0.4755	0.6545	0.7694
$x = 0.3$	0.1816	0.3455	0.4755	0.5590
$x = 0.4$	0.0955	0.1816	0.2500	0.2939

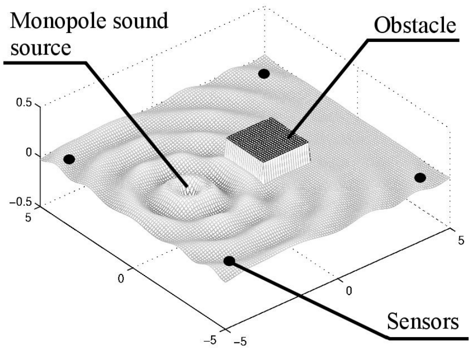

Fig. 2. Sound source and obstacle

VII. SOUND FIELD SIMULATOR APPLICATIONS

The modeling and analysis of a sound field were primarily developed as an initial tool to test whether it is possible to correctly detect and locate an an arbitrary object in a sound field, by only measuring the parameters of the sound field in a discrete number of points.

Fig. 2 illustrates a 2D sound field (membrane) with ideally reflective obstacle (3D object in sound field with reduced dimension), and Fig. 3 shows the readings of the four sensors (measurements) placed in the field. As one can see in Fig. 3, the readings without obstacle and with obstacle differ as expected.

Using more sensors (currently up to 14), and multilayer neural networks ([8]), the authors got encouraging results in detection of objects in a simulated sound field. A rectangular object can be detected with fair accuracy in the sound field, using neural networks. The analysis, and the creation of training and validation data sets, were made in the simulator. The simulator results, such as these on Fig. 3 can be used in the analysis of a sound field with objects. The goal is to try to establish usable relations between sensor readings and sound field geometry. Details, and more results from this area will be the subject of future technical papers.

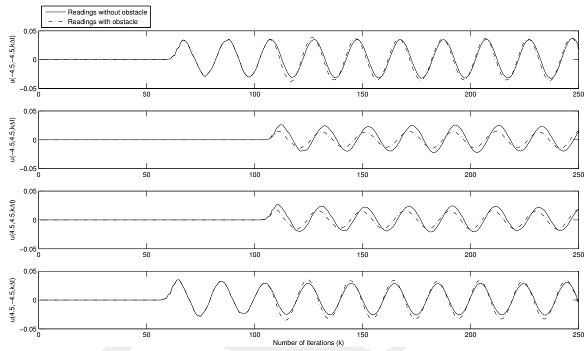

Fig. 3. Sensor readings - nominal readings without object in sound field (solid line), or with object in sound field (dashed line) .

The decribed simulator also allows simple defining and positioning sound sources, defining and positioning objects (through boundary conditions), and sensors for measuring sound field parameters.

VIII. CONCLUSIONS AND FUTURE WORK

It was shown that numerical approach in solving wave equation in sound field analysis can be used as a tool for investigating the possibility of object detection in a known sound field.

The simulator has a possibility of 3D simulation, but still doesn't have a usable 3D visualisation (only 2D layers), what is to be done in close future. Also, characteristics of sensors (microphones) are not yet considered, and sensors are considered ideal and unobtrusive which is not a real assumption.

Future work includes enhancing the simulator by adding more function (moving objects) and creating a more user-friendly interface. Physical verification of the simulator is also planned.

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